

Scalar gravitational waves and Einstein frame

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Abstract

The response of a gravitational wave detector to scalar waves is analysed in the framework of the debate on the choice of conformal frames for scalar-tensor theories. A correction to the geodesic equation arising in the Einstein conformal frame modifies the geodesic deviation equation. This modification is due to the

non-metricity of the theory in the Einstein frame, yielding a longitudinal mode that is absent in the Jordan conformal frame.

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Virtually every modern theoretical attempt to unify gravity with the remaining interactions requires the introduction of scalar fields (e.g. the dilaton and the moduli fields in string theory [1]).

In the literature on scalar fields it is claimed that the Jordan frame version of Brans-Dicke and scalar-tensor theories is untenable, owing to the problem of negative kinetic energies [2, 3]. In turn, the Einstein frame version of scalar-tensor theories – obtained by a conformal rescaling $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu}$ and a nonlinear field redefinition $\phi \rightarrow \tilde{\phi}$ – has a positive definite energy. In this frame however, there is a violation of the Equivalence Principle, due to an anomalous coupling of the scalar field ϕ to ordinary matter. Naturally, this violation is small and compatible with the available tests of the Equivalence Principle.¹ It is, indeed, a low-energy manifestation of compactified theories [5, 6, 7].

Acknowledging the debate in the literature about the conformal frames used for the description of scalar-tensor theories, we focus on possible phenomenologically distinctive features emerging in the conformal frames. As an example of such phenomena, we consider in this Letter the interaction of scalar waves with a gravitational wave detector, as seen in the Einstein frame. The same question has been addressed in the Jordan frame

¹For the consequences of such tests in a more specific framework, see e.g. [4].

[8].

If the scalar fields are required for unifying the fundamental interactions among particles at high energies, then they must be present in the early universe. Although a fundamental cosmological scalar field may have settled to a constant value during the era of matter domination [9], or even during inflation [10], it would leave an imprint in the cosmological background of gravitational waves by contributing spin zero modes.

At this point, the question arises, whether the gravitational wave detectors presently existing (i.e. the resonant detectors) or under construction (i.e. the interferometric detectors) can detect relic (cosmological) scalar gravitational waves². As we will see, a correction to the geodesic equation arises in the Einstein frame and modifies the geodesic deviation equation.

The main motivation for this analysis arises from cosmology: given the unavoidability of scalar fields in the early universe, scalar modes must be present in the relic gravitational wave background of cosmological origin, together with spin two modes. For simplicity, the prototype of scalar-tensor theories, i.e. Brans-Dicke [13] theory, is used in the following; however, the analysis below extends immediately to generalized scalar-tensor gravity.

²For a recent review of the status of gravitational wave detectors, see [11]; see also [12].

We plan our paper as follows. We begin by briefly discussing the essential features of the two conformal frames. Next, we compute the corrections to the geodesic and geodesic deviation equations that appear in the Einstein frame; finally, we conclude with a discussion of the results presented here and the comparison of our approach with the string formulation.

In the usual formulation of Brans-Dicke theory [13] in the so-called Jordan frame, gravitational waves are represented by the metric and scalar fields³

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad \phi = \phi_0 + \varphi , \quad (1)$$

where $\mathcal{O}(h_{\mu\nu}) = \mathcal{O}(\varphi/\phi_0) = \mathcal{O}(\epsilon)$, ϵ being a smallness parameter. The linearized field equations in a region outside sources are [15]

$$R_{\mu\nu} = T_{\mu\nu} [\varphi] + \mathcal{O}(\epsilon^2) = \frac{\partial_\mu \partial_\nu \varphi}{\phi_0} + \mathcal{O}(\epsilon^2) , \quad (2)$$

$$\square \varphi = 0 . \quad (3)$$

The energy density of the scalar waves seen by an observer with four-velocity u^μ is $\rho \equiv T_{\mu\nu}[\varphi]u^\mu u^\nu$ and its sign is indefinite: for example, for a monochromatic scalar wave $\varphi = \varphi_0 \cos(k_\alpha x^\alpha)$, one has $\rho = -(k_\alpha u^\alpha)^2 \varphi / \phi_0$.

³We adopt the notations and conventions of [14].

Brans-Dicke theory can be reformulated in the Einstein conformal frame [13] by means of the conformal transformation

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} , \quad \Omega = \sqrt{G\phi} , \quad (4)$$

and of the scalar field redefinition

$$\tilde{\phi} = \frac{1}{\chi} \ln \left(\frac{\phi}{\phi_0} \right) , \quad \chi \equiv \left(\frac{16\pi G}{2\omega + 3} \right)^{1/2} \quad (5)$$

where ω is the Brans-Dicke parameter. Scalar-tensor gravitational waves are described in the Einstein frame by

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu} , \quad \tilde{h}_{\mu\nu} = h_{\mu\nu} + \frac{\varphi}{\phi_0} \eta_{\mu\nu} , \quad \tilde{\phi} = \tilde{\phi}_0 + \tilde{\varphi} , \quad (6)$$

where one can use eq. (5) to express the field $\tilde{\varphi}$ through ϕ

$$\tilde{\varphi} = \frac{1}{\chi} \frac{\varphi}{\phi_0} . \quad (7)$$

The linearized field equations outside sources are⁴

$$\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} = 8\pi \left(\tilde{T}_{\mu\nu} [\tilde{\varphi}] + T_{\mu\nu}^{(eff)} [\tilde{h}_{\alpha\beta}] \right) , \quad (8)$$

⁴Note that the extension of our treatment and results to the most general scalar-tensor case is straightforward at this point, as the necessary modifications in the field equations only affect terms of higher order in the fields. Hence, within the linearized approximation adopted here, the non-geodesic term found below (eq. (13)) is the same for all scalar-tensor theories.

$$\square \tilde{\varphi} = 0 , \quad (9)$$

where

$$\tilde{T}_{\mu\nu} [\tilde{\varphi}] = \partial_\mu \tilde{\varphi} \partial_\nu \tilde{\varphi} - \frac{\eta_{\mu\nu}}{2} \partial^\alpha \tilde{\varphi} \partial_\alpha \tilde{\varphi} \quad (10)$$

and $T_{\mu\nu}^{(eff)}[\tilde{h}_{\alpha\beta}]$ is Isaacson's effective stress-energy tensor [16] (it yields the contribution of the tensor modes $\tilde{h}_{\alpha\beta}$ and is only of order ϵ^2). In the Einstein frame both scalar and tensor waves yield second order contributions to the effective energy density and the canonical form (10) of $\tilde{T}_{\mu\nu} [\tilde{\varphi}]$ (as opposed to $T_{\mu\nu} [\varphi]$ in eq. (2)) shows that the scalar contribution is non-negative (for a monochromatic plane wave $\tilde{\varphi} = \tilde{\varphi}_0 \cos(l^\alpha x_\alpha)$ one has $\tilde{\rho}_{\tilde{\varphi}} = \tilde{T}_{\mu\nu} u^\mu u^\nu = (l_\alpha u^\alpha \varphi)^2 \geq 0$).⁵ For this reason, the Einstein frame is often used to compute the energy density of the stochastic background of scalar waves (e.g. [8]); this procedure implicitly assumes that the Jordan and the Einstein frames are physically equivalent. If it was true, this equivalence would imply that the physics is the same in both frames, while we will see later that this is not the case for the geodesic deviation equation, which provides the theoretical ground for describing the response of a gravitational wave detector to a scalar wave.

Next, we set out to determine the corrections to the geodesic and geodesic deviation

⁵We do not address here the question of whether the reformulated theory is the physical one, as opposed to its Jordan frame counterpart. Our purpose is to obtain some phenomenological consequence of the Einstein reformulation of the theory, for the spectrum of scalar gravitational waves.

equations due to the reformulation of the theory to the Einstein frame. A convenient starting point is the conservation law for the matter energy-momentum tensor $\tilde{T}_{\mu\nu}^{(m)}$ in the Einstein frame [14]

$$\tilde{\nabla}_\mu \tilde{T}^{(m)\mu\nu} = -\frac{1}{\Omega} \frac{\partial \Omega}{\partial \phi} \tilde{T} \tilde{\nabla}^\mu \phi = -\frac{1}{2} \chi \tilde{T} \tilde{\nabla}^\mu \tilde{\phi}, \quad (11)$$

where $\tilde{T} = \tilde{T}^{(m)}_{\mu}{}^{\mu}$. For a dust fluid, i.e. for $T_{\mu\nu}^{(m)} = \rho u_\mu u_\nu$ (with $u_\alpha u^\alpha = -1$), one obtains the correction to the geodesic equation for a massive test particle

$$\frac{Dp^\mu}{D\lambda} = \frac{d^2 x^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = \frac{1}{2} \chi \partial^\mu \tilde{\phi}, \quad (12)$$

where $p^\mu \equiv dx^\mu/d\lambda$ is the four-tangent to the world line of a test particle. The correction is a force that couples universally to massive test particles (e.g. [5]); the equation of null geodesics instead is conformally invariant⁶ and is the same in the Jordan and in the Einstein frame.

A term with a formal similarity to our correction of the geodesic equation has been obtained also in a model inspired by string theory [18]. However, while the string dilaton in general couples differently to bodies with different internal nuclear structure, which

⁶The equation of null geodesics is obtained as the high frequency limit of the Maxwell equations, which are conformally invariant in four spacetime dimensions. Alternatively, for the Maxwell field one has $\tilde{T} = 0$ in eq. (11). In a different context, superconformal invariance was proposed to explain why newtonian gravity shadows cosmological constant effects [17].

carry a dilatonic charge [18], the Brans-Dicke field of scalar-tensor theories couples in the same way to every form of matter which has an energy-momentum tensor with a nonvanishing trace (cf. eq. (11)). In scalar-tensor theories there is no need to make assumptions on the form of the different couplings and on dilatonic charges, due to the universality of the coupling. See below for a comment on the distinct physical implications of our scalar-tensor case, with respect to the string inspired model of Ref. [15]

One can now derive the correction to the geodesic deviation equation; let $\{\gamma_s(\lambda)\}$ be a smooth one-parameter family of test particle worldlines (λ parameterizes the position along the worldline and s identifies curves in the family). If $p^\alpha = (\partial/\partial\lambda)^\alpha$, $s^\beta = (\partial/\partial s)^\beta$, one has $\partial p^\alpha/\partial s = \partial s^\alpha/\partial\lambda$ and $Dp^\alpha/Ds = Ds^\alpha/D\lambda$, where $D/Ds \equiv s^\alpha\nabla_\alpha$, $D/D\lambda \equiv p^\beta\nabla_\beta$. The relative acceleration of two neighbouring curves in $\{\gamma_s\}$ is found to be

$$\tilde{a}^\alpha = \tilde{R}_{\beta\delta\gamma}{}^\alpha s^\beta p^\gamma p^\delta + \frac{1}{2}\chi \frac{D}{Ds} (\partial^\alpha \tilde{\phi}) \quad (13)$$

(the calculation parallels the usual derivation of the geodesic deviation equation in general relativity – see e.g. [14]). A similar correction appears in string theory, but it depends on the dilatonic charge [18]. Note that the usual limit $\omega > 500$, which would make the contribution of scalar waves in eq. (13) small, does not apply in the Einstein frame. In fact, such a limit on ω assumes that the Jordan frame formulation of Brans-

Dicke theory is the relevant one, while we have abandoned it in favour of its Einstein frame counterpart.

From the geodesic deviation equation (13), we can calculate the time evolution of the separation Δx^i between two neighboring test particles. In the proper frame of one of them [16] one has

$$\Delta \ddot{x}^i = \tilde{R}_{j00}{}^i \Delta x^j + \frac{1}{2} \chi \frac{\partial^2 \tilde{\varphi}}{\partial x_i \partial x_j} \Delta x_j , \quad (14)$$

where the t -coordinate is the proper time of the particle at the frame's origin, and we use the notation $\dot{w} \equiv \partial w / \partial t$. For a plane wave propagating along the z -axis, choosing the gauge $\theta_{0\mu} = 0$, $\partial^\mu \theta_{\mu\nu} = 0$ (where $\theta_{\mu\nu} \equiv \tilde{h}_{\mu\nu} - 1/2 \eta_{\mu\nu} \tilde{h}_\alpha^\alpha + \tilde{\varphi} \eta_{\mu\nu}$), one obtains

$$\Delta \ddot{x}^i = \frac{1}{2} \left(\ddot{h}_{ij} + \delta_{ij} \chi \ddot{\tilde{\varphi}} \right) \Delta x^j . \quad (15)$$

For a purely scalar gravitational wave, i.e. for $h_{ij} = 0$, the unit matrix in this equation implies the existence of three oscillations, two transversal (for $i = j = 1, 2$) and one longitudinal (for $i = j = 3$), in the scalar sector of the fluctuations of the metric, which are gravitationally coupled to the detector, through the geodesic deviation equation.

The transverse modes are already present in the Jordan frame case, whereas the longitudinal oscillation is absent in the latter formulation of the theory (see [8]). The longitudinal mode arises due to the non-metricity of the Brans-Dicke theory in the Einstein frame, i.e. the fact that massive particles do not follow geodesics of the metric

$\tilde{g}_{\mu\nu}$. Notice that this longitudinal mode is not a gauge artifact, as in Ref. [19], but is a physical effect. Moreover, Eq. (15) shows that the longitudinal mode of the scalar waves is as important as the transverse scalar modes – the same coefficient appearing as a common multiplicative factor for all scalar modes.

It is interesting also to examine more closely the relation of our result with the string formulation [18]. There is a formal correspondence between the corrected geodesic deviation equation (14), and eq. (13) of Ref. [18], after the replacement $q \rightarrow \chi/2$ of the dilatonic charge q with the quantity $\chi/2$. However, the choice $q=0$ is always possible in the model of Ref. [18]. This corresponds to the only possibility to have, within that model, a universal dilaton coupling. All other values of the dilaton coupling to matter depend on the composition of the test particles. Hence, two different gravitational wave detectors, made of distinct materials, would respond differently to a scalar gravitational wave. Setting a vanishing coupling q in Ref. [18] yields no physical effect. On the other hand, in our case there is no way to get rid of the (composition independent) physical effect by any choice of coupling, as χ cannot be set to zero.

Irrespectively of whether the Einstein frame formulation is to be preferred to its Jordan counterpart or not, one is motivated at least by its occurrence in the literature in looking at the effect of scalar gravitational waves in the Einstein version of the theory. It is a fact – and our result shows it clearly – that in the latter there is a longitudinal

effect associated with a scalar wave, whereas such an effect is absent in the Jordan frame formulation. From the phenomenological point of view, we must account also for the necessary smallness of the violations of the Equivalence Principle in the Einstein frame formulation.

The main conclusion emerging from our work is that the Einstein frame description of the interaction of a scalar gravitational wave with a gravitational wave detector differs from the Jordan frame picture [8], as made clear also from the string theory analysis of Ref. [18]. This fact testifies of the physical inequivalence of the two conformal frames.

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